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Localization of gravity waves on a channel with a random bottom

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We present a theoretical study of the localization phenomenon of gravity waves by a rough bottom in a one-dimensional channel. After recalling localization theory and applying it to the shallow-water case, we give the first study of the localization problem in the framework of the full potential theory; in particular we develop a renormalized-transfer-matrix approach to this problem. Our results also yield numerical estimates of the localization length, which we compare with the viscous dissipation length. This allows the prediction of which cases localization should be observable in and in which cases it could be hidden by dissipative mechanisms.

1. Introduction

In this paper, we study surface gravity waves propagating on a one-dimensional channel with a random bottom. The case of periodic bottoms has been studied in Davies & Heathershaw (1984) and Mei (1985). Various results on hydrodynamics with random media or boundaries are reviewed in Mysak (1978). In the case of random bottoms, Guazzelli, Guyon & Souillard (1983) suggested that the phenomenon known as Anderson localization in solid-state physics could be observed on shallow waves: in this case Anderson localization implies that a periodic plane wave of wavelength λ coming on to the part of the channel with a random bottom will eventually be totally reflected. More precisely, the amplitude of the disturbance created by the wave dies off exponentially with distance, with a typical length ξ, called the localization length. Experiments to observe this phenomenon are being carried out in Marseille at the Laboratoire de Physique des Systèmes Désordonnés by M. Belzons, E. Guazzelli and O. Parodi. The random bottoms that we study in the present paper are of the same type as those used in these experiments; this will enable a comparison between theoretical predictions and experimental results.

The paper deals only with linear theories and is organized in two parts. Another paper (Devillard & Souillard 1986) discusses, for a nonlinear wave equation in a disordered medium, how nonlinearities may modify localization.

First, in §2, after recalling the basic elements of localization theory, we discuss the shallow-water theory on a rough bottom – a series of random steps – and we estimate the localization lengths for various wavelengths λ. In the asymptotic regimes where $\lambda \rightarrow \infty$ and $\lambda \rightarrow 0$, the behaviour of localization lengths is derived.

Since the predictions of the shallow-water theory in the regime $\lambda \rightarrow 0$ are clearly wrong, we then use in §3 the full linear potential theory. Localization of eigenstates has not been proved in such a case, so we have to make approximations. We have chosen two approximations which are valid independently of the wavelength, and are convenient not only for numerics, but also for some analytical results. We first assume that the random steps have a mean length much greater than the height of water. This assumption is reminiscent of the 'wide-spacing approximation' (Newman 1965; Srokosz & Evans 1979), although we do not have a restriction on the wavelength. The second approximation is Miles (1967) approximate variational solution to scattering by one step. Within this framework, we have then been able to estimate the localization length and also to obtain satisfactory asymptotic behaviour for it as $\lambda \rightarrow \infty$ and $\lambda \rightarrow 0$.

In §4, we discuss some physical phenomena not taken into account within the full linear potential theory, phenomena which could hide localization or limit the range of validity of our theoretical study. Viscosity can be a major effect, but we show that there is a range of wavelengths for which localization should be observable in a tank experiment such as the one in Marseille.

Since the full linear potential theory, and even the validity of the linearization, have a scale invariance, the localization length will scale accordingly. Thus our results also apply at oceanographic scales.

The experimental results obtained currently in Marseille are in good agreement with the results of the present paper. A comparison of the theoretical and experimental conclusions will be presented in a forthcoming letter (Belzons *et al.* 1988*a*). A detailed description of the experimental set-up and of the corresponding results can be found in Belzons, Guazzelli & Parodi (1988*b*); see also Guazzelli (1986), Devillard (1986).

2. Shallow-water theory

We consider a one-dimensional channel and look at the propagation of a gravity wave. We shall denote by η the elevation of the free surface of the liquid due to the travelling of the wave, and by h the height of the channel. If the wavelength λ is much larger than the depth, and if relative fluctuations of height are not too large, the propagation of a surface gravity wave is described by the shallow-waves equation

$$\partial_x(h\, \partial_x \eta) = \frac{g^{-1}\, \partial^2 \eta}{\partial t^2}. \tag{1}$$

Since we will be interested in permanent regimes, we take the Fourier transform with respect to time of (1) and obtain

$$g\partial_x(h(x)\, \partial_x \Psi) = -\omega^2\, \Psi \tag{2}$$

where

$$\Psi = \int_{\mathbb{R}} \eta(x,t)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}t.$$

The problem we want to study is the behaviour of the solutions of (1) when the bottom of the channel is rough, that is, $h(x)$ is a random function. In that case, and for a very large class of random bottoms, a remarkable phenomenon happens: for almost every realization of the bottom, all the proper modes of the stationary equation (2) are exponentially localized in space owing to the randomness of the bottom; that is, the proper modes decay exponentially with distance for $x \rightarrow \pm\infty$, in direct contrast to the situation for a flat or a periodic bottom. This phenomenon is also closely related to the following one: if we look at the transmission $t_\omega(L)$ of a plane wave of frequency ω by a part of a channel of length L with a rough bottom, this transmission decays exponentially with L in the absence of any dissipation in the medium. If one considers the time-dependent problem, the localization of the proper modes implies that for any square integrable initial wave, the energy of the wave outside a finite region of space remains small *uniformly in time* for a large enough region. The so-called Lyapunov exponent for (2) turns out to be a quantity of peculiar interest: it is defined as the asymptotic rate of exponential increase of a solution of the Cauchy problem of the stationary equation, namely for given ω

$$\gamma = \lim_{x \to +\infty} (2x)^{-1} \log \left(|\Psi(x)|^2 + |k(x)^{-1}\, \Psi'(x)|^2\right), \tag{3}$$

where $$k(x) = \omega (gh(x))^{-\frac{1}{2}}.$$

The exponent γ is a natural measure of the rate of increase (or decrease) of an oscillatory function; it must be noted that, given any initial condition $(\Psi(0),\ \Psi'(0))$, the same limit γ in (3) is attained for almost every realization of the bottom (i.e. with probability one in the choice of the bottom); e.g. choose an arbitrary pattern of height h_0 on the interval $[0,\ 1]$, and reproduce it on each interval $[n,\ n+1]$ with a height h_n chosen at random independently of the other heights. The 'almost every' proviso excludes exceptional choices of $\{h_n\}_{n\in\mathbb{Z}}$ which for example happen to be periodic in n. If the pattern is varied, the convergence in (3) will work even better. Note also that γ is independent of the initial condition $(\Psi(0),\ \Psi'(0))$.

It turns out that for a very large class of random bottoms an extension (Kotani 1982; Minami 1986) of a well-known theorem of Fürstenberg (1963) ensures that γ is positive for almost every ω. In turn, this implies (Delyon, Simon & Souillard 1987) exponential localization of all proper modes for almost every realization of the potential. In addition, the Lyapunov exponent turns out to be the rate of exponential decay of the proper modes (i.e. they behave as modulo oscillations as $e^{-\gamma|x|}$ for $x \to \pm\infty$); the transmission coefficient $t_\omega(L)$ of a disordered section of channel of length L decreases also as $e^{-2\gamma L}$ as $L \to \infty$. The inverse ξ of the Lyapunov exponent γ is a characteristic length which is naturally called the localization length.

In fact the localization phenomenon was first discovered by Anderson (1958) in solid-state physics when studying Schrödinger equations with a random potential, or more precisely discrete equations known as 'tight binding' models. In the context of classical physics, localization was first discussed by Hodges (1982). For reviews and references to localization theory, we refer to Thouless (1979), Souillard (1987), Simon & Souillard (1984), Lee & Ramakrishnan (1985). In fact localization is a very general phenomenon for linear wave equations in a random medium (Souillard 1987) and since the shallow-waves equation (1) with random height has features very similar to the Schrödinger equation with random potential, localization theory can be transposed to it with minimal adaptation.

Since it is known from the theory that the qualitative features of the problem do not depend on the details of the model, we have chosen for the present study the simplest random bottom allowing comparison with experiments on shallow waves: the random bottom is a stepwise function, the length s_i of the ith step being chosen at random, independently of the others and uniformly in the interval $[s_{\min},\ s_{\max}]$, and the height h_i being chosen with uniform distribution in $[h_{\min},\ h_{\max}]$. In some cases, we take the lengths of the steps to be constant, that is $s_{\min} = s_{\max}$ (see figure 1).

We shall set
$$H = \tfrac{1}{2}(h_{\max} + h_{\min}), \quad \Delta h = \tfrac{1}{2}(h_{\max} - h_{\min}),$$
$$S = \tfrac{1}{2}(s_{\min} + s_{\max}), \quad \Delta s = \tfrac{1}{2}(s_{\max} - s_{\min}).$$

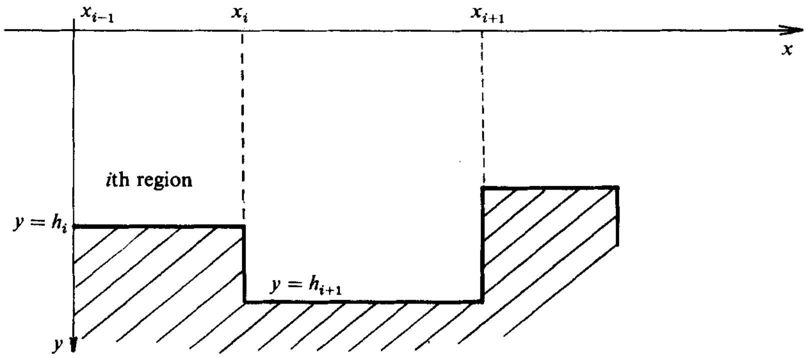

FIGURE 1. Random bottom made of a succession of rectangular steps of random heights.

Under these conditions, the continuity of the surface elevation η and the continuity of the flux at points x_i imply that

$$\partial_x \Psi|_{x_i+\epsilon} = \frac{h_i}{h_{i+1}} \partial_x \Psi|_{x_i-\epsilon}, \tag{4a}$$

$$\Psi(x_i+\epsilon) = \Psi(x_i-\epsilon). \tag{4b}$$

Since the equations can be integrated on each step, the solution can be easily reconstructed from the solution at points x_i. Thus from now on we shall denote $\Psi(x_i-\epsilon)$ by Ψ_i and we set $\Lambda_i = -h_i^{\frac{1}{2}} \boldsymbol{k}_i^{-1} \partial_x \Psi|_{x=x_i-\epsilon}$ where $\boldsymbol{k}_i$ is the wave vector corresponding to frequency ω for a plane wave on a flat channel of depth h_i, namely $\boldsymbol{k}_i = \omega(gh_i)^{-\frac{1}{2}}$. We thus have

$$(\Psi_{i+1}, \Lambda_{i+1})^{\mathrm{t}} = \boldsymbol{M}_{i+1}(\Psi_i, \Lambda_i)^{\mathrm{t}}$$

with a transfer matrix $\boldsymbol{M}_i = \boldsymbol{S}_i^{-1} \boldsymbol{R}_{\theta_i} \boldsymbol{S}_i$ and where $\boldsymbol{R}_{\theta_i}$ is the rotation matrix of angle $\theta_i = \omega(x_i - x_{i-1})(gh_i)^{-\frac{1}{2}}$ and $\boldsymbol{S}_i$ is the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & H^{\frac{1}{2}}/h_i^{\frac{1}{2}} \end{bmatrix}.$$

We shall denote

$$\boldsymbol{W}_i = (\Psi_i, \Lambda_i)^{\mathrm{t}}.$$

We have indicated above that all the proper modes of (1) are exponentially localized in space for almost every realization of the bottom (Delyon *et al.* 1987). We thus can turn to the evaluation of the Lyapunov exponent γ, and hence of the localization length $\xi = \gamma^{-1}$. For a given ω, the Lyapunov exponent γ is also given for almost every vector $\boldsymbol{W}_0$, and almost every realization of the bottom by

$$\gamma = \lim_{N\to\infty} X_N^{-1} \log \left(\Big\| \prod_{0\leqslant j\leqslant N-1} \boldsymbol{M}_{N-j} \boldsymbol{W}_0 \Big\|\right), \tag{5}$$

where X_N is the absciss of the point where $\boldsymbol{W}_N$ is calculated (if the lengths of the steps are deterministic, $X_N = NS$; otherwise for large N, $X_N/N \sim S$). This formula is particularly convenient for numerical calculations: a vector $\boldsymbol{W}_0$ is chosen, $\boldsymbol{W}_N = \boldsymbol{M}_N \boldsymbol{M}_{N-1} \ldots \boldsymbol{M}_2 \boldsymbol{M}_1 \boldsymbol{W}_0$ is calculated and, if N is sufficiently large, we observe that $(2N)^{-1} \log(\| \boldsymbol{W}_N \|^2)$ tends to a positive constant; typically one has to go to lengths X_N of the order of a few times γ^{-1} to see convergence on the computer results. This method will be used later on to obtain estimates of the Lyapunov exponent. However, we first discuss another estimate of the Lyapunov exponent which will allow us to derive analytically its asymptotics in the regimes $\omega \to 0$ or $\omega \to \infty$.

2.1. *Asymptotics of the localization length for small and large frequency; use of the invariant measure*

The first asymptotic behaviour we can look for is that at long wavelength; it is well known (see Papanicolaou 1978) that in this regime

$$\gamma(\omega) \sim \omega^2 \quad \text{for } \omega \to 0.$$

The second asymptotic behaviour we can look for is that for small wavelength, which is discussed now and for which we shall get

$$\gamma(\omega) \sim \text{const} \quad \text{for } \omega \to \infty.$$

In order to establish this behaviour, we use the fact that the Lyapunov exponent can be written as

$$\gamma = \lim_{N\to\infty} X_N^{-1} \sum_{0 \leqslant i \leqslant N-1} \log\left(\frac{\|\boldsymbol{M}_{i+1} \boldsymbol{W}_i\|}{\|\boldsymbol{W}_i\|}\right),$$

where

$$\boldsymbol{W}_i = \boldsymbol{M}_i \boldsymbol{M}_{i-1} \ldots \boldsymbol{M}_1 \boldsymbol{W}_0$$

and we shall take advantage of the fact that, the orientation of the vector $\boldsymbol{W}_i$ depending on the realization of the $\boldsymbol{M}_i$, the probability distribution of its orientation will converge to a probability measure μ_{inv} when i becomes large. If the matrices $\boldsymbol{M}_i$ are statistically independent, it is known (Fürstenberg 1963) that

$$\gamma = \int (2s)^{-1}\,\mathrm{d}\tau \int \log\left(\|\boldsymbol{M}(\tau)\,\boldsymbol{U}_\phi\|^2\right) \mathrm{d}\mu_{\text{inv}}(\phi), \tag{6}$$

where $\boldsymbol{U}_\phi$ is the unit vector with angle ϕ with respect to the x-axis, $\boldsymbol{M}(\tau)$ is the transfer matrix, τ denoting the set of random parameters. In our case, $\mathrm{d}\tau$ is simply the product $\mathrm{d}\mu_h\,\mathrm{d}\mu_s$, the probability distribution of having a step with height h and length s. The main interest of (6) lies in the possibility of finding approximate analytical solutions for the invariant measure and, from them, approximate analytical solutions for the asymptotic behaviour of γ, as developed below.

We thus look for an approximation of the invariant measure for small $\lambda (\lambda \ll s_{\text{min}})$. As λ/S goes to zero, θ_i is always much larger than 2π. As soon as h_i is allowed to fluctuate a little ($\Delta h/H \gg \lambda/S$), the values of θ_i will cover the whole interval $[0, 2\pi]$ approximately with uniform density, even if s is deterministic. More precisely, let us take a vector $\boldsymbol{W}_i$ with angle $(0x, \boldsymbol{W}_i) = \phi_i$, and consider the probability distribution $\mu_{\phi_{i+1}}$ of $\phi_{i+1} = (0x, \boldsymbol{W}_{i+1})$, h_i being fixed h_{i+1} and $(x_{i+1} - x_i)$ being allowed to vary according to the probability $\mu(\tau)$. We claim that $\mu_{\phi_{i+1}}$ will be roughly the image of the uniform Lebesgue measure on $[0, 2\pi]$ by the matrix $\boldsymbol{S}_i^{-1}$ (in Devillard 1986, the above assertion is checked numerically and the agreement is good). If we apply (6), we obtain for $\omega \to \infty$

$$\begin{aligned}\gamma(\omega = \infty) = 2^{-1}\left[\int s^{-1}\,\mathrm{d}\mu(s)\right]&\left\{\iint \mathrm{d}\mu(h)\,\mathrm{d}\mu(h') \iint (2\pi)^{-2}\,\mathrm{d}\theta\,\mathrm{d}\psi\right.\\ &\times\left[\log\left((\cos\theta\cos\psi - \left(\frac{h}{h'}\right)^{\frac{1}{2}}\sin\theta\sin\psi)^2 + (h^{\frac{1}{2}}\sin\theta\cos\psi + h'^{\frac{1}{2}}\cos\theta\sin\psi)^2\right.\right.\\ &\left.\left.\left. - \log\cos^2\psi + h'\sin^2\psi\right)\right]\right\}.\end{aligned}$$

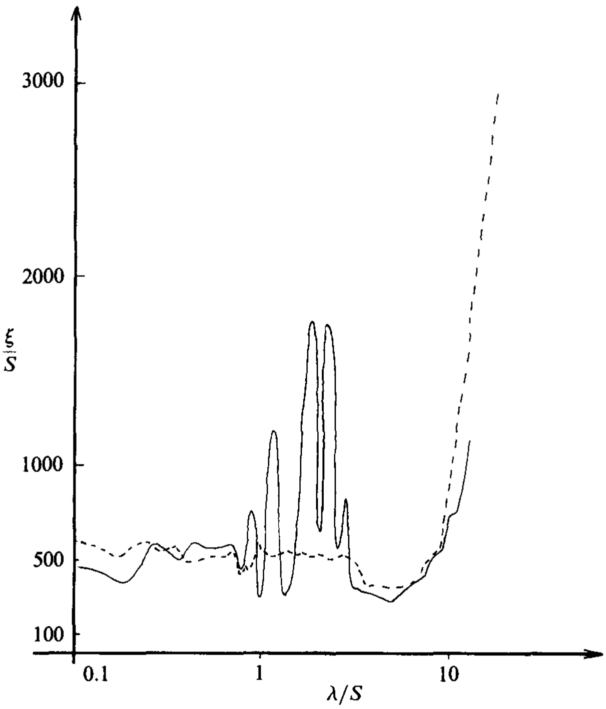

FIGURE 2. ξ/S as a function of λ/S (shallow-water theory). The values of the parameters of our model are: $\Delta h/H = 0.3$, $\Delta s/S = 0.1$ for the continuous curve, and $\Delta s/S = 0.5$ for the dashed curve.

Using

$$\int_{0\leqslant x\leqslant 2\pi} \log(1+a\sin x+b\cos x)\,\mathrm{d}x = 2\pi\,[\log(1+(1-a^2-b^2)^{\frac{1}{2}})-\log 2]$$

we find

$$\gamma(\omega=\infty) = \langle s^{-1}\rangle \left\langle\!\left\langle \log 2^{-1}\left(1+\left(\frac{h}{h'}\right)^{\frac{1}{2}}\right)\right\rangle\!\right\rangle, \tag{7}$$

where the double brackets denote the average with respect to the distribution of h and h'. One can show, in particular in view of §3 below, that the above formula is exact in the limit $\lambda \rightarrow 0$ and $H/\lambda \rightarrow 0$ with

$$H\left[1+\log\left(\frac{h_{\text{max}}}{h_{\text{min}}}\right)\right] \ll \lambda \ll S\frac{\Delta h}{H}.$$

If the disorder on the heights of the steps is weak (i.e. $\Delta h \ll h_{\text{min}}$), we have

$$\gamma \sim \frac{1}{48}\left(\frac{\Delta h}{H}\right)^2 \langle s^{-1}\rangle.$$

The above result for $\gamma(\omega=\infty)$ can also be derived using the non-independent matrices $\boldsymbol{R}_{\theta_i}\boldsymbol{S}_i(\boldsymbol{S}_{i-1})^{-1}$.

2.2. *Numerical results for the localization length*

We have calculated numerically the localization length $\xi = 1/\gamma$ when s is randomly chosen in $[S-\Delta s, S+\Delta s]$. When $(\Delta s/s)$ is small, we observe resonances, which tend to disappear as $(\Delta s/S)$ increases (see figure 2). This corresponds to the fact that for

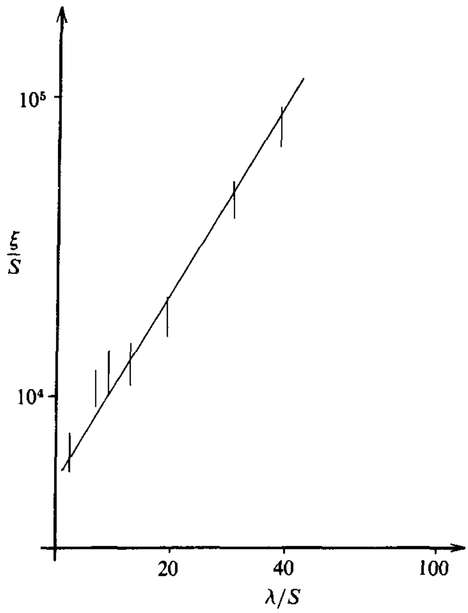

FIGURE 3. Log–log plot of ξ/S as a function of λ/S. (The slope of the straight line is 2.) Here s is taken deterministic and $\Delta h/H = 0.1$.

$\Delta s/S$ small, the lengths of the steps correspond almost to a periodic situation: although the heights can be random, a certain amount of periodicity in the direction of the x-axis remains. These resonances are then a reminder of the effects of passing bands in periodic structures.

On figure 2, the change of behaviour from small to large ξ occurs typically at $\lambda \sim 6S$; simulations not represented on the figure show that this change of behaviour happens at larger λ when disorder increases. Beyond this crossover value of λ, the waves become very ill localized. At large λ, we have checked the $(\omega^{-2} = \lambda^2)$-dependence of the localization length which was mentioned in the previous subsection (see figure 3).

On figure 2, we see also that ξ goes to a constant at small λ, in good agreement with the results of the previous subsection (see figure 4).

Of course, the above behaviour of the localization length at small wavelengths holds only within the shallow-water theory, but is unreasonable from the hydrodynamical point of view: this will be corrected in §3, where we shall handle the study of localization within the full potential theory.

Other physical limitations to the theory such as the effects of nonlinearities, surface tension and viscosity will be discussed in §4.

3. The full potential theory on a random bottom

The shallow-wave approximation is no longer valid when the wavelength λ is comparable with or smaller than the depth of the channel. Even in the case of large wavelength, when relative fluctuations of height of the channel are important, shallow-water theory is not valid (cf. (14) below). Therefore, we study the full linearized potential theory.

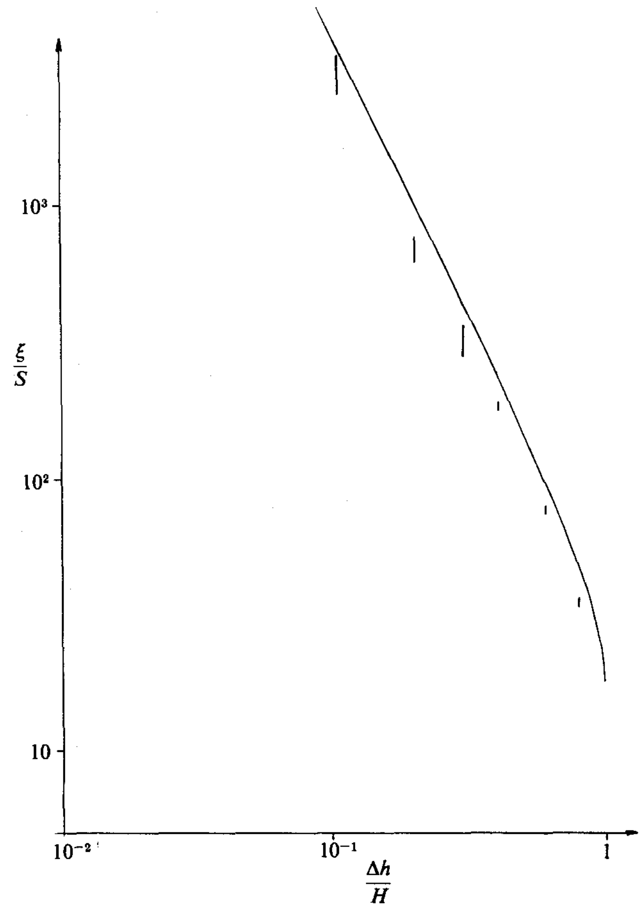

FIGURE 4. ξ/S as a function of $\Delta h/H$, with $\lambda/S \ll 1$. Here $\Delta s/S = 0.5$. The continuous curve represents the analytic formula (7), the vertical bars denote the numerical estimates together with their errors.

Denoting by v the velocity, we have: $v = \mathrm{Re}\,(-\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{grad}\,\Phi)$, Φ being the velocity potential. The equation for Φ is still $\Delta\Phi = 0$ with boundary conditions:

$$\partial_y \Phi + \omega^2 g^{-1} \Phi = 0 \quad \text{on } y = 0,$$

$$\frac{\partial \Phi}{\partial n} = 0 \quad \text{at the bottom.}$$

The elevation η of the free surface is then given by

$$\eta = \mathrm{Re}(-\mathrm{i}\omega g^{-1}\Phi\,\mathrm{e}^{-\mathrm{i}\omega t}).$$

For such a system, when the height of the bottom is random as in §2, there are, up to now, no results on the possible localization of proper modes, nor on the behaviour of the transmission; we develop such a theory here for the first time, using a method of renormalized transfer matrices. It will allow us to show localization of all proper modes in a natural situation and also to get numerical results on the localization length as well as analytical predictions for various of its asymptotics.

We first recall some standard results; the experts can go directly to the discussions after (11) and (14).

3.1. *Linear potential theory on a piecewise-constant bottom*

As in §2, we shall consider a piecewise-constant bottom. Denoting by Φ_i the velocity potential in the ith region, one has

$$\left.\begin{aligned} \Delta\Phi_i &= 0 \quad \text{for } x_{i-1} < x < x_i \quad (0 < y < h_i), \\ \partial_y \Phi_i + \omega^2 g^{-1}\Phi_i &= 0 \quad \text{for } x_{i-1} < x < x_i \quad (y = 0), \\ \partial_y \Phi_i &= 0 \quad \text{for } x_{i-1} < x < x_i \quad (y = h_i), \end{aligned}\right\} \tag{8}$$

together with the matching conditions

$$\left.\begin{aligned} \Phi_i &= \Phi_{i-1} \quad \text{for } x = x_{i-1} \quad (0 < y < \mathrm{Min}\,(h_i, h_{i-1})), \\ \partial_x \Phi_i &= 0 \quad \text{for } x = x_{i-1} \quad (h_{i-1} < y < h_i\,(\text{if } h_{i-1} < h_i)), \\ \partial_x \Phi_i &= \partial_x \Phi_{i-1} \quad \text{for } x = x_{i-1} \quad (0 < y < \mathrm{Min}\,(h_i, h_{i-1})). \end{aligned}\right\} \tag{9}$$

In each region of constant depth, the solution can be expanded over a complete set of explicit solutions:

$$\Phi_i(x, y) = (A_i\,\mathrm{e}^{\mathrm{i}kx} + B_i\,\mathrm{e}^{-\mathrm{i}kx})\,\chi_i(y) + \sum_{\kappa_n} D_i(\kappa_n)\,\mathrm{e}^{-\kappa_n(x-x_{i-1})}\,\phi_i(\kappa_n, y) + \sum_{\kappa_n} C_i(\kappa_n)\,\mathrm{e}^{\kappa_n(x-x_i)}\,\phi_i(\kappa_n, y), \tag{10}$$

where $\mathrm{i}k$ and κ_n $(n = 1, 2, \ldots)$ are solutions to the dispersion relation

$$\kappa \tan(\kappa h_i) = -K, \tag{11}$$

$$K = \omega^2 g^{-1}$$

and $\quad \chi_i(y) = F(\mathrm{i}k, y),\, \phi_i(\kappa_n, y) = F(\kappa_n, y)$

with $\quad F(\kappa, y) = \surd 2(h_i - K^{-1}\sin^2 \kappa h_i)^{-\frac{1}{2}} \cos(\kappa(h_i - y)).$

The problem of linear wave propagation on a one-dimensional random bottom can now be formulated as follows: we have two propagating modes, described by the coefficients A_i and B_i in (10), coupled to infinitely many non-propagating modes, described by the coefficients $D_i(\kappa_n)$ and $C_i(\kappa_n)$, $n = 1, 2, \ldots$; the coupling comes from the matching conditions (9).

A possible approach to this problem would be to truncate the sum over κ_n at some order N, and study products of random $(2N+2)\times(2N+2)$ matrices. One could then in principle compute $N+1$ positive Lyapunov exponents; the smallest of these, which is obtained last, will be the inverse of the localization length. Such an approach would be very unwieldy and lengthy numerically if N is not small.

The approach that we have chosen instead is based on the following observation: the non-propagating modes decay uniformly with wavelength, away from the edges of the steps:

$$\mathrm{e}^{-\kappa_n|x-x_{i-1}|} < \mathrm{e}^{-(\pi/2h_i)|x-x_{i-1}|} \quad \forall\, n. \tag{12}$$

In view of this, we have restricted our attention to random bottoms such that the lengths of the steps are much greater than the depth of water:

$$|x_i - x_{i-1}| \gg h_i. \tag{13}$$

We can then assume that the non-propagating modes originating at one step are negligible when they reach the next step. This is like the wide-spacing approximation of Srokosz & Evans (1979), who assume, instead of (13), that the distance between two obstacles is large compared with the wavelength.

We still have infinitely many modes coupled on the edge of each step, just as in the much-studied case of a single step, or shelf. Here we have chosen to incorporate Miles' approximate variational solution (Miles 1967), which is a good approximation at all wavelengths. As we are going to use the explicit form of this solution, we first recall briefly how it is obtained.

3.2. *The transfer matrix for a shelf in Miles' theory*

The height of water is h_1 for $x > 0$ and $h_2 > h_1$ for $x < 0$; the horizontal component of the velocity, denoted $U(x, y)$, satisfies

$$U \sim (\mathrm{i}\boldsymbol{k}_1 A_1 \mathrm{e}^{\mathrm{i}k_1 x} - \mathrm{i}\boldsymbol{k}_1 B_1 \mathrm{e}^{-\mathrm{i}k_1 x})\,\chi_1(y) \quad \text{as}\, x \to +\infty,$$
$$U \sim (\mathrm{i}\boldsymbol{k}_2 A_2 \mathrm{e}^{\mathrm{i}k_2 x} - \mathrm{i}\boldsymbol{k}_2 B_2 \mathrm{e}^{-\mathrm{i}k_2 x})\,\chi_2(y) \quad \text{as}\, x \to -\infty.$$

At $x = 0$, $U(0, y)$ must be linear in $(A_1 + B_1)$ and $(A_2 + B_2)$:

$$U(0, y) = (A_1 + B_1)\, u_1(y) + (A_2 + B_2)\, u_2(y) \quad (0 < y < h).$$

Now the approximation, which is supported by a variational argument, is that the unknown functions $u_1(y)$ and $u_2(y)$ are both taken proportional to $\chi_1(y)$. Miles then solves exactly for the scattering matrix and obtains:

$$\begin{bmatrix} -B_1 \\ A_2 \end{bmatrix} = (1 + N^2 - \mathrm{i}X) \begin{bmatrix} N^2 - 1 - \mathrm{i}X & -2\left(\dfrac{\boldsymbol{k}_2}{\boldsymbol{k}_1}\right)^{\frac{1}{2}} N \\ -2\left(\dfrac{\boldsymbol{k}_1}{\boldsymbol{k}_2}\right)^{\frac{1}{2}} N & 1 - N^2 - \mathrm{i}X \end{bmatrix} \begin{bmatrix} -A_1 \\ B_2 \end{bmatrix},$$

where

$$\left.\begin{aligned} N &= 2K(\boldsymbol{k}_1 \boldsymbol{k}_2)^{\frac{1}{2}} \sinh(k_2(h_2 - h_1))\,(\boldsymbol{k}_1^2 - \boldsymbol{k}_2^2)^{-1} \\ &\qquad (Kh_1 + \sinh^2 \boldsymbol{k}_1 h_1)^{-\frac{1}{2}} (Kh_2 + \sinh^2 \boldsymbol{k}_2 h_2)^{-\frac{1}{2}}, \\ X &= 4K^2 \boldsymbol{k}_1 (Kh_1 + \sinh^2 \boldsymbol{k}_1 h_1)^{-1} \sum_n \kappa_{2n} (\boldsymbol{k}_1^2 + \kappa_{2n}^2)^{-2} \\ &\qquad \sin^2(\kappa_{2n}(h_2 - h_1))\,(Kh_2 - \sin^2 \kappa_{2n} h_2)^{-1} \end{aligned}\right\} \quad (14)$$

In order to use Miles' result in our problem, we need a transfer matrix rather than a scattering matrix. We also replace (A_i, B_i) by the following more physical coefficients in our context:

$$\boldsymbol{\Psi}_i = (A_i \mathrm{e}^{\mathrm{i}k_i x_i} + B_i \mathrm{e}^{-\mathrm{i}k_i x_i})\,\chi_i(0),$$
$$\boldsymbol{\Omega}_i = -\boldsymbol{k}_i^{-1} \frac{\mathrm{d}\boldsymbol{\Psi}_i}{\mathrm{d}x_i}.$$

We then have, taking into account both scattering at step x_i and propagation between x_i and x_{i+1},

$$(\boldsymbol{\Psi}_{i+1}, \boldsymbol{\Omega}_{i+1})^{\mathrm{t}} = \boldsymbol{R}_{\theta_i} \boldsymbol{M}_i (\boldsymbol{\Psi}_i, \boldsymbol{\Omega}_i)^{\mathrm{t}},$$

where $\boldsymbol{R}_{\theta_i}$ is the matrix of rotation by an angle $\theta_i = \boldsymbol{k}_{i+1}(x_{i+1} - x_i)$ and

$$\boldsymbol{M}_i = \frac{\chi_{i+1}}{\chi_i} \left(\frac{\boldsymbol{k}_i}{\boldsymbol{k}_{i+1}}\right)^{\frac{1}{2}} \begin{bmatrix} N_i^{J_i} & \dfrac{-X_i}{N_i} \\ 0 & N_i^{-J_i} \end{bmatrix},$$

where $\chi_i = \chi_i(y = 0) = \surd 2(h_i + K^{-1} \sinh^2 \boldsymbol{k}_i h_i)^{-\frac{1}{2}} \cosh(\boldsymbol{k}_i h_i)$,

$$J_i = \begin{cases} -1 & \text{if}\, h_{i+1} \geqslant h_i, \\ +1 & \text{if}\, h_{i+1} < h_i, \end{cases}$$

and N_i and X_i are given by (14), where the smaller of h_i and h_{i+1} must play the role of h_1 and the larger the role of h_2.

The scale invariance of the initial equations has of course been preserved throughout: the matrices $\boldsymbol{R}_{\theta_i}$ and $\boldsymbol{M}_i$ are invariant under

$$\begin{aligned} h_i &\rightarrow \tau h_i, \\ \boldsymbol{k}_i &\rightarrow \tau^{-1}\boldsymbol{k}_i \quad (\text{and } \omega^2 \rightarrow \tau^{-1}\omega^2 \text{ in the dispersion relation}), \\ K &\rightarrow \tau^{-1}K, \\ x_{i+1}-x_i &\rightarrow \tau(x_{i+1}-x_i). \end{aligned}$$

The results presented below will therefore be displayed in a scale-invariant form, in terms of λ/S, H/S, $\Delta h/H$, $\Delta S/S$ and ξ/S, where H is the mean height, λ the corresponding wavelength, S the mean length of a step and ξ the localization length.

Note that in the limit of large wavelengths, the matrix $\boldsymbol{M}_i$ reduces to

$$\boldsymbol{M}_i = \begin{bmatrix} 1 & -X\left(\dfrac{h_i}{h_{i+1}}\right)^{\frac{1}{4}}\left(\dfrac{h_D}{h_S}\right)^{\frac{1}{4}} \\ 0 & \left(\dfrac{h_i}{h_{i+1}}\right)^{\frac{1}{2}} \end{bmatrix},$$

where $h_D = \operatorname{Max}(h_i, h_{i+1})$ and $h_S = \operatorname{Min}(h_i, h_{i+1})$.

As λ gets very large, X is of the order of $\boldsymbol{k}_S h_S(\log(\boldsymbol{k}_S h_S)^{-1} + \log(h_D/h_S))$. The off-diagonal term in $\boldsymbol{M}_i$ will therefore be negligible when

$$\boldsymbol{k}_D h_D\left(\log(\boldsymbol{k}_D h_D)^{-1} + \log\left(\frac{h_D}{h_S}\right)\right) \ll 1. \tag{15}$$

This 'shallow-water' condition may be simplifed as

$$k_D h_D\left(1 + \log\left(\frac{h_D}{h_S}\right)\right) \ll 1.$$

Therefore, for a given disorder and large λ, the matrix $\boldsymbol{M}_i$ reduces to

$$\begin{bmatrix} 1 & 0 \\ 0 & \left(\dfrac{h_i}{h_{i+1}}\right)^{\frac{1}{2}} \end{bmatrix},$$

which is the matrix obtained from the shallow-water theory in the basis we have chosen.

3.3. *Products of renormalized transfer matrices and localization of waves in the full potential theory*

As seen from the two previous subsections, the study of waves on a random bottom in the full potential theory can be associated to a problem of products of random 2×2 renormalized transfer matrices. The situation is now very reminiscent of the shallow-water case: the Lyapunov exponent associate with the products of such random matrices will again be positive and again from this we can see that all proper modes are exponentially localized; furthermore the localization length can be numerically computed as the inverse of the Lyapunov exponent associate with the products of

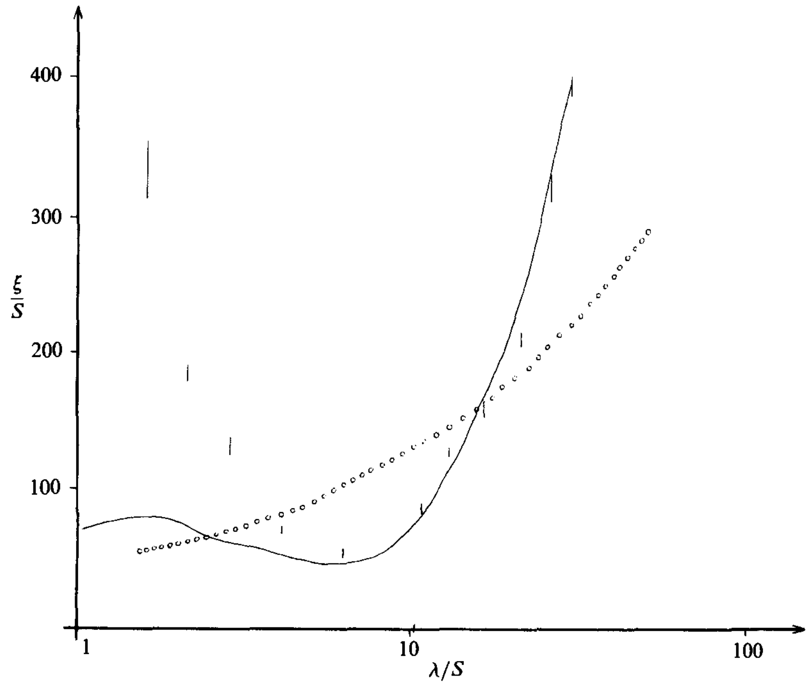

FIGURE 5. ξ/S as a function of λ/S. Here $\Delta s/S = 0.5$, $\Delta h/H = 5/7$, $H/S = 7/16$. The continuous curve corresponds to the shallow-water theory, the bars to the numerical simulations for the full potential theory; the dotted line corresponds to a crude estimate of the viscous dissipation length $l_{\nu,0}$ when the scale is fixed by $S = 4$ cm.

the renormalized transfer matrices and we can deduce the behaviour of the localization length in various asymptotics. At fixed disorder, for large λ, shallow-water theory is of course recovered. However, at small λ, the asymptotic behaviour is now very different from that of the shallow-water theory: it can be shown that the localization length diverges exponentially in the asymptotic limit $\lambda \rightarrow 0$ (see the Appendix) as

$$\frac{\xi}{S} \sim \langle \exp\{-4\omega^2 g^{-1} h\}\rangle^{-1}$$

instead of going to a constant as predicted by the shallow-water theory. Physically, this high-frequency regime can be understood as follows: a surface wave creates a disturbance in the velocity potential which is mainly localized in a strip of height typically g/ω^2, near the surface, if $g/\omega^2 \ll h$. In deep water, the disturbance dies off exponentially and thus, as ω increases, the wave will be less and less sensitive to the fluctuations of the bottom.

We can use our approach to obtain numerical estimates of the localization length. The results of the numerical simulations are presented in figure 5. The shallow-wave theory and our full potential theory are in good agreement, as expected, for large λ and differ significantly for small λ, as predicted theoretically above.

In figure 5, the viscous dissipation length as evaluated in §4 below is also displayed in order to allow a discussion of the possible observation of localization (see §4).

4. Possible observation of localization and physical limitations

In order to ascertain qualitatively whether these localization effects can be seen experimentally, we must discuss the physical phenomena that could hide localization or, at least, limit the range of applicability of the theory developed above.

In an experimental set-up such as that in Marseille (see Belzons *et al.* 1988*b*; Guazzelli 1986), three physical aspects that we have not taken into account are: the effects of viscosity; the limitations of the linear theory due to the presence of nonlinear terms; the effects of surface tension.

4.1. *The effects of viscosity*

In the linear theory used above, we have completely neglected the effect of dissipation; clearly dissipation implies exponential decay of transmission. Since one of the most natural ways of observing localization would be to observe the exponential decay of transmission with length, a strong attenuation due to dissipation would make it difficult to separate the attenuation due to localization. The effect of localization would not be hidden by dissipation if the localization length is shorter or of the order of the viscous dissipation length l_ν, whereas if $l_\nu \ll \xi$ localization effects may be hidden by the dissipation effects. It is thus important for our purpose to estimate the viscous dissipation length.

A comparison of the dissipation length and the localization length at large wavelength for a particular model can be found in Akkermans & Maynard (1984), but this is a regime for which localization would be, in any case, unobservable since the localization length is there very large.

To get an estimate of the viscous dissipation length for all reasonable wavelengths, we use the full potential theory for a laminar flow (note however that for abrupt steps, the flow will not be laminar near the bottom). For a laminar flow, there are two different mechanisms of viscous dissipation: dissipation in the bulk of the fluid and dissipation near the bottom and the edges of the channel (Landau & Lifschitz 1971).

For a channel with a flat bottom of depth h and width b, the dissipation length is

$$l_\nu = \frac{1}{\gamma_\nu(h)},$$

where

$$\gamma_\nu(h) = U^{-1}\left[2\nu k^2 + (\tfrac{1}{2}\nu\omega)^{\frac{1}{2}}\left\{\frac{k}{\sinh(2kh)} + b^{-1}\right\}\right]$$

and U is the group velocity,

$$U = 2^{-1} g^{\frac{1}{2}}(k\tanh(kh))^{-\frac{1}{2}}\left[\tanh(kh) + \frac{kh}{\cosh^2(kh)}\right].$$

For large wavelengths, dissipation takes place mainly near the bottom and the edges of the channel and γ_ν takes the simple form (for $b \gg h$)

$$\gamma_\nu(h) \sim 2^{-1}\left(\frac{\nu\pi}{\lambda}\right)^{\frac{1}{2}} g^{-\frac{1}{4}} h^{-\frac{5}{4}}.$$

For short wavelengths, dissipation takes place mainly in the bulk and γ_ν takes the form

$$\gamma_\nu(h) \sim 4\nu g^{-\frac{1}{2}}(2\pi/\lambda)^{\frac{5}{2}}.$$

We obtain a rough estimate of the inverse of the viscous dissipation length on a stepwise-constant bottom by averaging γ_ν with respect to h:

$$\gamma_{\nu,0} = (2\Delta h)^{-1} \int_{h_{\min} \leqslant h \leqslant h_{\max}} \gamma_\nu(h)\, \mathrm{d}h,$$

$$l_{\nu,0} = \frac{1}{\gamma_{\nu,0}}.$$

On figure 5 in §3 above, the dotted line represents $l_{\nu,0}/S$ as a function of λ/S, with $S = 4$ cm.

The essential conclusion that can be drawn from this figure is that there is a significant range of wavelengths for which the localization length is shorter or of the order of the viscous dissipation length. We can thus predict that localization will be observable in this range of wavelengths.

Notice also that the viscous dissipation length, which is not scale-covariant, should increase faster than linearly with the scale. Therefore larger scales should be easier for the observation of localization

4.2. *Limits of the linear theory*

On a flat bottom, the weakly nonlinear theory gives Stokes waves. The first nonlinear corrections to the linear potential theory are negligible there when

$$k\eta \ll 1, \quad \eta h^{-1} \ll 1, \quad k^{-2}h^{-3}\eta \ll 1. \tag{16}$$

In the Marseille experiment, $\eta \sim 1$ mm is compatible with the above restrictions, and the observed nonlinear corrections are at most a few per cent. This suggests that the linear theory, on such bottoms as we consider, has a certain stability with respect to small nonlinearities satisfying the local conditions (16).

An interesting question is certainly the study of the joint effect of localization and nonlinearities; some results on the transmission problem for the nonlinear Schrödinger equation with a random potential have been obtained recently (Devillard & Souillard 1986).

4.3. *Surface tension*

When λ is very small, the effects of surface tension also become relevant. The wavelength is to be compared with the capillarity length of the fluid $a = (2\alpha/(\rho g))^{\frac{1}{2}}$ where ρ is the volumic mass of the fluid and α the surface tension for the air–fluid interface ($a \approx 3.9$ mm for water). Neglecting surface tension gives a relative error $2\pi^2 a^2 \lambda^{-2}$ when solving the dispersion relation for k or λ. This error will be negligible for

$$\lambda \gg \surd 2\,\pi a \approx 1.7 \text{ cm}.$$

5. Conclusion

We have studied in this paper the phenomenon of Anderson localization for water waves, first with the shallow-water wave equation, then using the full potential theory and finally discussing and estimating the various phenomena limiting their possible experimental observation. The main conclusion is the prediction of a range of wavelengths for which the localization phenomenon will be observable.

Appendix

In this Appendix we derive the asymptotics of the localization length in the short-wavelength regime in the framework of the renormalized-transfer-matrices approach developed in this paper. In order to simplify the notation, we set $\Psi_{+,i} = \Psi(x_i+\epsilon)$, and we use

$$(\boldsymbol{\Psi}_+, \boldsymbol{\Omega}_+)^{\mathrm{t}}_{i+1} = \frac{\chi_{i+2}}{\chi_{i+1}}\left(\frac{\boldsymbol{k}_{i+1}}{\boldsymbol{k}_{i+2}}\right)^{\frac{1}{2}} \begin{bmatrix} N^J & \dfrac{-X}{N} \\ 0 & N^{-J} \end{bmatrix} \boldsymbol{R}_\theta(\boldsymbol{\Psi}_+, \boldsymbol{\Omega}_+)^{\mathrm{t}}_i.$$

Since $\theta = \boldsymbol{k}_i s$, and since $\boldsymbol{k}_i \to \infty$ as $K \to \infty$, we may approximate the invariant measure by the Lebesgue measure. We neglect the correlations between the matrices and we shall perform the calculation for the case where the lengths of the steps s are deterministic. Then we have

$$\gamma = (2S)^{-1} \int \mathrm{d}\mu(\tau) \int_{0 \leqslant \theta \leqslant 2\pi} \mathrm{d}\theta/2\pi \log\left\{\left(N^J \cos\theta - \frac{X}{N}\sin\theta\right)^2 + (N^{-J}\sin\theta)^2\right\},$$

where τ stands for the set of all random variables on which the matices depend. When the height of the steps is drawn with uniform probability between $h_{\min}$ and $h_{\max}$, we have

$$\int \mathrm{d}\mu(\tau) = \iint_{\substack{h_{\min} \leqslant h_1 \leqslant h_{\max} \\ h_{\min} \leqslant h_2 \leqslant h_{\max}}} \frac{\mathrm{d}h_1\,\mathrm{d}h_2}{4\Delta h^2}$$

from which we obtain

$$\begin{aligned}\gamma = (4S)^{-1} \Bigg\{ &\iint_{\substack{h_{\min} \leqslant h_1 \leqslant h_{\max} \\ h_{\min} \leqslant h_2 \leqslant h_1}} (4\Delta h^2)^{-1}\,\mathrm{d}h_1\,\mathrm{d}h_2 \\ &\times \int_{0 \leqslant \theta \leqslant 2\pi} \mathrm{d}\theta/2\pi \log\left[\left(N\cos\theta - \frac{X}{N}\sin\theta\right)^2 + (N^{-1}\sin\theta)^2\right] \\ &+ \int_{\substack{h_{\min} \leqslant h_2 \leqslant h_{\max} \\ h_{\min} \leqslant h_1 \leqslant h_2}} (4\Delta h^2)^{-1}\,\mathrm{d}h_1\,\mathrm{d}h_2 \\ &\times \int_{0 \leqslant \theta \leqslant 2\pi} \mathrm{d}\theta/2\pi \log\left[\left(N^{-1}\cos\theta - \frac{X}{N}\sin\theta\right)^2 + (N\sin\theta)^2\right]\Bigg\}.\end{aligned}$$

Now, N and X can be readily expressed as functions of h_{D} and h_{S}, $h_{\mathrm{D}} = \mathrm{Max}\,(h_1, h_2)$, $h_{\mathrm{S}} = \mathrm{Min}\,(h_1, h_2)$ (N and X are symmetric in h_1 and h_2). Using

$$\int_{0 \leqslant x \leqslant 2\pi} \log(1 + a\sin x + b\cos x)\,\mathrm{d}x = 2\pi\left[\log\left(1 + (1 - a^2 - b^2)^{\frac{1}{2}}\right) - \log 2\right]$$

if $a^2 + b^2 < 1$, we have

$$\begin{aligned}&\int_{0 \leqslant \theta \leqslant 2\pi} \mathrm{d}\theta/2\pi \log\left\{\left(N\cos\theta - \frac{X}{N}\sin\theta\right)^2 + (N^{-1}\sin\theta)^2\right\} \\ &\qquad = \int_{0 \leqslant \theta \leqslant 2\pi} \mathrm{d}\theta/2\pi \log\left\{\left(N^{-1}\cos\theta - \frac{X}{N}\sin\theta\right)^2 + (N\sin\theta)^2\right\}.\end{aligned}$$

Since the cases $J = 1$ and $J = -1$ occur with the same probability, we obtain

$$\gamma = (2S)^{-1} \iint_{\substack{h_{min} \leqslant h_2 \leqslant h_{max} \\ h_{min} \leqslant h_1 \leqslant h_2}} (4\Delta h^2)^{-1}\, dh_1\, dh_2$$

$$\times \left\{ \log\left\{\frac{1}{2}\left[N^2 + \frac{1+X^2}{N^2}\right]\right\} + \log\left\{\frac{1}{2}\left[1 + \left(1 - \left[2X\Big/\left(N^2 + \frac{1+X^2}{N^2}\right)\right]^2 \right.\right.\right.\right.$$

$$\left.\left.\left.\left. - \left[N^2 - \frac{1+X^2}{N^2}\right]^2 \left[N^2 + \frac{1+X^2}{N^2}\right]^{-2}\right)^{\frac{1}{2}}\right]\right\}\right\}.$$

We are now left with the problem of calculating N and X. Let us first define $\lambda = (k_D/k_S)^{\frac{1}{2}}$. We have

$$\lambda N = \sinh(k_D(h_D - h_S))\,(k_S - k_D)^{-1} \sinh^{-1}(k_D h_D) \sinh^{-1}(k_S h_S)\,(k_S + k_D)^{-1}\, 2Kk_D$$

$$\times \left(1 + \frac{Kh_S}{\sinh^2 k_S h_S}\right)^{-\frac{1}{2}} \left(1 + \frac{Kh_D}{\sinh^2 k_D h_D}\right)^{-\frac{1}{2}}.$$

Let us set now

$$\alpha = \sinh(k_D(h_D - h_S))\,(k_S - k_D)^{-1} \sinh^{-1}(k_D h_D) \sinh^{-1}(k_S h_S)$$

$$= K^{-1} \frac{\sinh(k_D(h_D - h_S))}{\sinh(k_D h_D - k_S h_S)}.$$

A. 1. *Calculation of* $k_D - K$ *and* $k_S - K$

We set $k_D h_D = a$, $Kh_D = b$, $\phi(x) = x \tan hx$. We have

$$b = \phi(b) + (a-b)\phi'(b) + \frac{(a-b)^2}{2\phi''(\theta)}, \quad \theta \in [a, b];$$

thus

$$b(1 - \tanh b) = (a-b)\,[\tanh b + b(1 - \tanh^2 b)] + (a-b)^2\,(1 - \tanh^2\theta)\,(1 - \theta\tanh\theta).$$

Neglecting the last term, we obtain

$$k_D - K = 2K[e^{-2Kh_D} + (1 - 4Kh_D)\, e^{-4Kh_D} + \dots].$$

We then check that $(a-b)^2\,(1 - \tanh^2\theta)\,(1 - \theta\tanh\theta)$ is at least of order e^{-5Kh_D}.

A. 2. *Calculation of* α

Using the previous expressions for $k_D - K$, we have

$$\alpha = \frac{K^{-1}(1 - e^{-2k_D(h_D - h_S)})\Xi}{1 - e^{-2k_D(h_D - h_S)}\Xi^2},$$

with $\Xi = e^{(k_S - k_D)h_S} = 1 + \epsilon + v + \dots$,

where $$\epsilon = 2Kh_S(e^{-2Kh_S} - e^{-2Kh_D})$$

$$v = 2Kh_S[(1 - 4Kh_S)\, e^{-4Kh_S} - (1 - 4Kh_D)\, e^{-4Kh_D}] + 2(Kh_S)^2\,(e^{-2Kh_S} - e^{-2Kh_D})^2.$$

By expanding the previous formula for α, we now obtain

$$\alpha = \Xi K^{-1}\,[1 + e^{-2k_D(h_D - h_S)}\,(\Xi^2 - 1) + e^{-4k_D(h_D - h_S)}\,\Xi^2\,(\Xi^2 - 1) + \dots]$$

$$= K^{-1}\,(1 + \epsilon + v)\,[1 + (2\epsilon + 2v + 4\tau)\, e^{-2k_D(h_D - h_S)} + o(e^{-4k_D(h_D - h_S)})]$$

with $\tau = (Kh_S)^2\,(e^{-2Kh_S} - e^{-2Kh_D})^2$. We thus have

$$\alpha \sim \frac{\Xi}{K} \sim K^{-1}\{1+2Kh_S[e^{-2Kh_S}+e^{-2Kh_D}]\}+O((Kh_S)^2\,e^{-4Kh_S}),$$

$$(1+Kh_S\,\sinh^{-2}(k_S h_S))^{-\frac{1}{2}} = 1-2Kh_S\,e^{-2Kh_S}+O((Kh_S)^2\,e^{-4Kh_S}),$$

$$(k_S+k_D)^{-1} = (2K)^{-1}(1-e^{-2Kh_S}-e^{-2Kh_D}) = O((Kh_S)^2\,e^{-4Kh_S}),$$

$$\left(\frac{k_S}{k_D}\right)^{\frac{1}{2}} = 1+e^{-2Kh_S}-e^{-2Kh_D}+O((Kh_S)^2\,e^{-4Kh_S}),$$

$$N = 1+O((Kh_S)^2\,e^{-4Kh_S}),$$

$$X = 4K^2\,k_S(Kh_S+\sinh^2(k_S h_S))^{-1}\,\Sigma,$$

with

$$\Sigma = \sum_{\substack{k,\ \text{non-propagated modes}\\ \text{corresponding to } h_D}} k(k_S^2+k^2)^{-2}\,(Kh_D-\sin^2 kh_D)^{-1}\,\sin^2(k(h_D-h_S)),$$

$$X \sim 16\,K^3\,e^{-2Kh_S}\,\Sigma \sim e^{-2Kh_S},$$

$$N^2+N^{-2} = 2+O((Kh_S)^4\,e^{-8Kh_S}).$$

Hence, the behaviour of γ:

$$\gamma \sim \langle s^{-1}\rangle \iint_{\substack{h_{min}\leqslant h_D\leqslant h_{max}\\ h_{min}\leqslant h_S\leqslant h_D}} (4\Delta h^2)^{-1}X^2\,dh_S\,dh_D,$$

$$\gamma \sim \langle s^{-1}\rangle\langle e^{-4Kh}\rangle.$$

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